Exploring Measurement Error with Cookies: A Real and Virtual Approach via Interactive Excel

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Experimental measurement is a vital component of the scientific method; and interpreting the error in a measurement is also very important to grasp. Collecting data to understand the behavior of a system leads to developing mathematical models, which has been recognized as a crucial process for students to learn in mathematics (AMATYC, 2006; NCTM, 2000) and the sciences (Siebert and McIntosh, 2001). We present a simple, guided-inquiry investigation using stacked cookies to develop a simple linear mathematical model and to explore measurement error by incorporating errors as part of the investigation. The model is then investigated further by exploring an interactive Excel simulation. A conceptual understanding is developed by hands-on manipulation and virtual experimentation. Numerous higher-order thinking and science process skills are used throughout the investigation. This activity has usefulness in a variety of mathematics classes, as well as the sciences, and illustrates the elegant connection between both disciplines.

Experimental Approach

Is there a relationship between the height of a stack of sandwich cookies and the number of cookies? The mode of questioning used to explore with students is embedded in the discussion. Draw a line and place 11 circles in a row by tracing around a cookie (see Figure 1). Make the first circle dashed and label it zero. Number the remaining circles 1 to 10. This will form a number line.
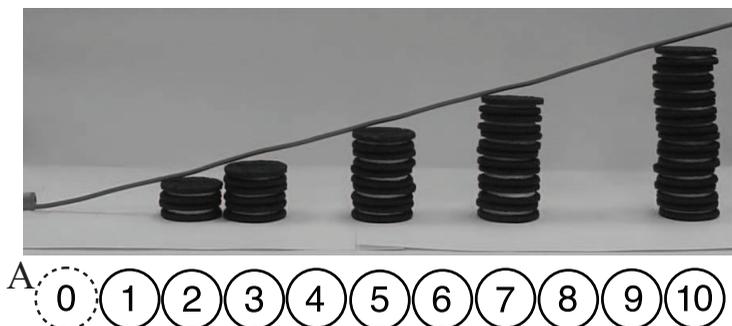

Figure 1: Cookie Number Line

Using sandwich cookies and a centimeter ruler, measure, to the nearest millimeter, the height of a variety of stacks (0, 2, 3, 5, 7, and 10 cookies) by placing them on the line and recording the data. The stacking of the cookies on the number line also gives a visual bar-like graph as well. With a string placed at point A (Figure 1) and held taut along the stacks, the linear nature of the cookie stacks can be discovered visually too. The collected data can be entered in a graphing calculator or spreadsheet software, such as Excel, and a linear regression performed on the data. Here is an equation, written in terms of the variables studied, generated using regular Oreo™ cookies:

$$\text{Height } = 1.24(\text{number of cookies}) - 0.078 \qquad \text{with } r^2 = 0.9998$$

What does the slope represent? What are the units of the slope? The slope is the average thickness of the cookies in centimeters. What are the heights of 20 and 150 cookies? Here you need to use the regression equation to make predictions. How could you verify your predictions? What can you conclude about the uniformity of the thickness of the cookies? The regression equation shows that the cookies are very consistent in their thickness. How would the graph change if Double-Stuf Oreo™ cookies were used? Could you draw this graph with only one measurement? This plot would have a greater slope equivalent to the average thickness of Double-Stuf

Oreo™ cookies, which could be determined by measurement of 10 (or any number) Double-Stuf Oreo™ cookies and finding the average thickness (1.5 cm/cookie). This part of the activity is presently used in intermediate and college algebra during Peer-Led Team Learning (PLTL) workshops. More on the PLTL workshop method can be found at http://www.pltl.org.

Now we are going to repeat the procedure with the introduction of two different errors. Case I: repeat the stacking of cookies as above; however, place a poker chip under each and every cookie as you stack them. Measure the heights, record, and perform a linear regression for Case I. Compared to the original equation, how is the regression equation different for this error? Case II: place one cookie on the circles of the number line and then stack the appropriate number of cookies as in the original experiment (no poker chips). Measure the stack heights for each stack and include the extra cookie in this measurement but do not count it in the number. Record all data and perform a linear regression for Case II. Again, compared to the original equation, how is the regression equation different for this error? The results that students discover are summarized in Table 1. Future uses include a PLTL workshop in Calculus II on error estimation in Simpson's rule and the Trapezoidal rule for numerical integration. Understanding how errors arise in physical applications will help the students to understand the error in these tools, and why we would wish to have a bound on the error.

Table 1: Three Experimental Trials

Case	Slope	Intercept	Error
Original	Average thickness	Should be zero	Random error
Error Case I	Increased slope by thickness of poker chip	Should be zero	Proportional error
Error Case II	Average thickness	Increased by one cookie thickness	Constant error

If you mixed the Double-Stuf Oreo™ cookies with the regular Oreo™ cookies and used them indiscriminately in the stacks, how would it influence the graph? Suppose your ruler had 2 mm of space before the zero mark. How would the graph be influenced? Suppose your ruler started at 0.3 cm and not zero. What would the error look like? Suppose that every centimeter on the ruler was 0.9 cm instead of 1.0 cm. What effect would this have? These are the types of questions that probe the understanding of the concepts explored in this investigation. Using an interactive Excel spreadsheet allows us to investigate these questions more quickly without further experimentation.

Virtual Simulation in Excel

The interactive Excel spreadsheet, which consists of eight worksheets, is available to download at http://academic.pgcc.edu/~ssinex/cookies_error.xls. The basics of constructing an interactive computational spreadsheet are discussed in Sinex (2004). This spreadsheet requires the Analysis ToolPak to be loaded (the last tab gives instructions to do this). The graphs and data tables respond to changes in a number of variables by either entering numerical values (yellow colored cells) or adjusting sliders on scroll bars or clicking on the arrows. The tabs, shown in Figure 2, allow you to navigate from worksheet to worksheet. The discussion that follows is keyed to the various tabs on the Excel spreadsheet. We recommend opening the Excel spreadsheet and examining it as you read. Comment boxes, denoted by the little red triangle in the upper right corner of the cell, give an explanation of calculations being done when the cursor is placed on the cell.

Figure 2: Tabs on Excel Workbook

The Data Analysis tab allows you to enter the stack height measurements and a regression is done automatically as data is entered. The analysis is done in a quick fashion and early in the class with minimal explanation of regression. Predictions can also be easily explored. How would the graph change if all the cookies were thinner? The slope of the graph can be changed and compared to the regression line of the entered data. The behavior of the line can be explored for thinner or thicker cookies. Discovering the slope as the average thickness of the cookies is reinforced.

On the Error Types tab, the constant, proportional and random errors can be adjusted to see how the graph responses. The random errors, the noise in the data, are generated using the RANDBETWEEN function available in Excel. The data points generated by an error can be varied and are compared to the ideal stack data of average thickness cookies. The three types of errors are illustrated as separate graphs in Figure 3. How the behavior of the graphs changes should have been discovered earlier; however, now the magnitude of the error can be explored. Both the numerical and graphical aspects can be compared.

The Differences tab calculates and plots the difference for the various errors. Figure 4 illustrates the nature of the errors where the squares show the constant
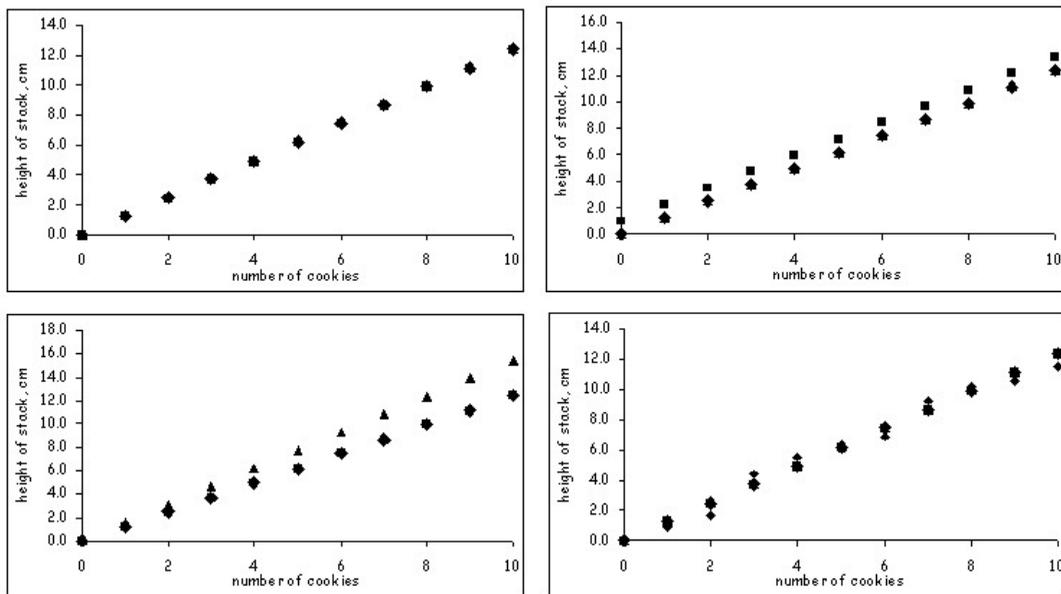
Figure 3: Error Types (upper left—ideal stack, upper right—constant error, lower left—proportional error, lower right—random error)

error, the triangles show the proportional error, and the diamonds show the random error or scatter in the data.

The slope of the proportional difference is the spacer thickness or error. Thus the slope of the proportional error graph (Figure 3) is the combination of the slope of the stack height versus number of cookies graph (or average thickness) plus the error. The scatter in the random error is why the R^2 value is less than one and decreases as random error increases. The random error is probably due mostly to variation in the actual thickness of the cookies and any measurement error using the ruler.

The errors that have been explored are summarized in Table 2 and also under the Discussion tab. Students discover the errors and their behavior through the real and virtual experimentation. Experimentally, random error can be enhanced by measuring the height of stacks of chocolate chip cookies, since they do not have a consistent thickness.

The Extension tab demonstrates the combination of random and systematic (constant and proportional) errors. This is the real-world of measurement and is illustrated in Figure 5. Students are asked to predict how the graph will respond
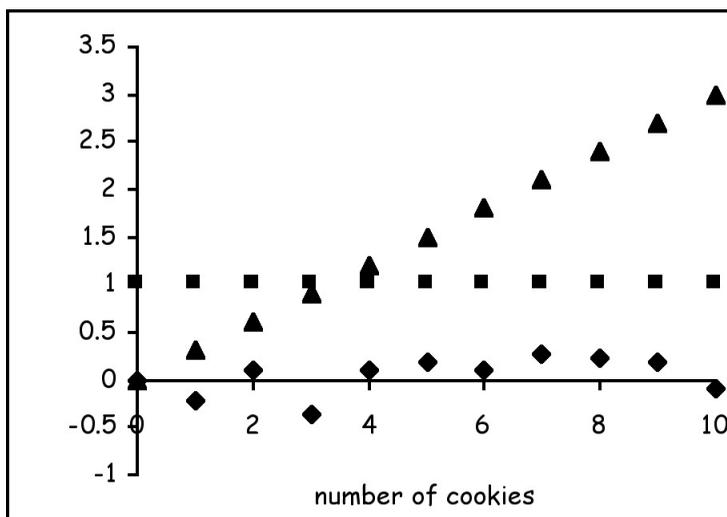
Figure 4: Difference Plot (Key: squares = constant, triangles = proportional, diamonds = random)

when the errors are combined. Then they can explore the effect of the size of the random error. An excellent follow-up question to check understanding is "How would the graphs change if the systematic error was corrected?" Could a small systematic error be masked by random error?

Table 2: Types of Error

Type of Error		Mathematical effect on graph	Experimental options
Random		Causes scatter in data, $r^2 < 1$	can be minimized by replicate measurements—no control over
Systematic	Constant	Shifts intercept	can be discovered and corrected or controlled, such as
	Proportional	Changes the slope	mis-calibrated measuring devices

Some Final Thoughts

Measurement error is a very important concept for students to grasp. Developing a mind-set to consider it for all measurements is an excellent habit for a scientist. For a detailed discussion of data analysis see Kirkup (2002). Discovering error, espe-
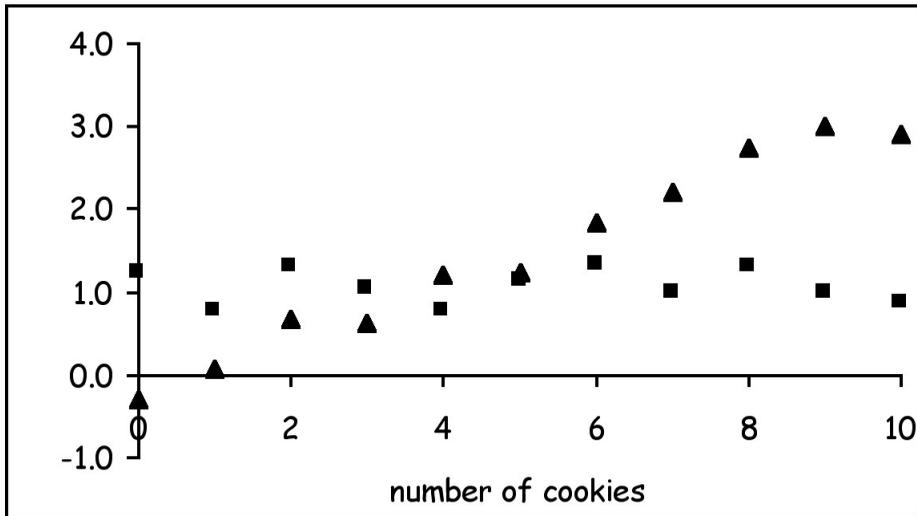
Figure 5: Combined Error (Key: squares = constant plus random, triangles = proportional plus random)

cially systematic error, can be done by plotting two different measurement techniques or devices and examining their symmetry about the $x = y$ line as shown in Figure 6 (see Comparison tab). Random error would scatter the data around the $x = y$ line, while systematic error shifts the intercept and/or the slope. From the graph and some interpolating, measurements taken on Device 2 would be greater than those on Device 1. Deciding which one is correct requires a standard to assess accuracy. Try plotting the data from the cookies in the comparison format to discover the errors.

The activity supports the "rule of four" by connecting the numerical, graphical, symbolic, and communications aspects and strengthens the use of mathematical language in a scientific analysis.

References

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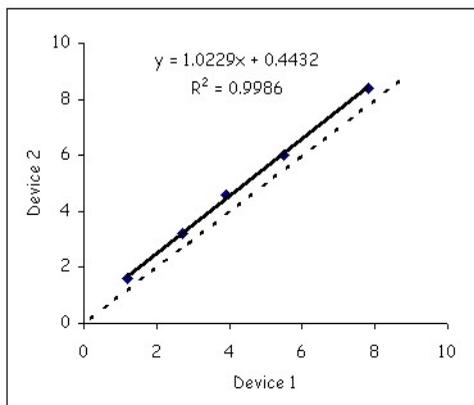
Figure 6: Comparison Plot (dashed line is $x = y$ line)

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Lucky Larry #85

Larry wants to find the natural number for which the product of the next two consecutive natural numbers is twenty.

Larry's solution was as follows:

$$(x + 2)(x + 1) = 20, \text{ then cancel out a two}$$
$$x(x + 1) = 10, \text{ and multiply by } x$$
$$x^2 + 1 = 10, \text{ subtract one}$$
$$x^2 = 9,$$

so that $x = 3$.

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